CONTINUOUS–TIME INCENTIVES IN HIERARCHIES

Emma Hubert[1],

with the relevant ideas and useful advice of Dylan Possamaï[2].

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[1]LAMA, Université Paris–Est Marne–la–Vallée, France.
[2]IEOR Department, Columbia University, USA.

MOTIVATION

A hierarchy. In an organisation, a hierarchy usually consists of a power entity at the top with subsequent levels of power underneath.

- Dominant structure in our contemporary society.
- Raises many questions, on its efficiency, its cost, its optimal size...
- First mathematical models: Williamson (1967), Calvo and Wellisz (1978) and Keren and Levhari (1979).

Incentives in a hierarchy. Related with Principal–Agent problems, to model information asymmetries within a hierarchy: ex–ante (adverse selection) or ex–post (moral hazard) the signing of contracts between the entities of the hierarchy (Stiglitz (1975) and Mirrlees (1976)).

- Succession of nested Stackelberg equilibria.
- On moral hazard: Laffont (1990), Yang (1995)...
- Discrete–time models, mostly consisting of a single period.

THE ONE–PERIOD MODEL

Sung (2015) – Pay for performance under hierarchical contracting.

- A Manager is hired by a Principal to subcontract with n Agents. Each worker (Manager and Agents) controls his own output process, and all outputs are assumed to be independent.
- A one–period model: 'For ease of exposition and without loss of generality, we formulate a discrete–time model which is analogous to its continuous–time counterpart' (Sung (2015)).
- Extending the reasoning of Holmström and Milgrom (1987), Sung restricts the study to linear contracts, and states that this restriction is 'without loss of generality, as long as our results are interpreted in the context of continuous–time models as in Holmström and Milgrom (1987)' (Sung (2015)).

- A hierarchical Principal–Agent model in one–period with moral hazard.

The Principal (she) is risk–neutral and represents the shareholders (or the investors) of a firm.

The Agents are the $n+1$ risk–averse workers of the firm (with CARA utility). Each Agent $i \in \{0, \dots, N\}$ (he) produces the random outcome X^i by carrying out his own task:

$$X^i = \alpha^i + \sigma^i W^i,$$

where $W^i \sim \mathcal{N}(0,1)$ are i.i.d.

The effort of the i–th Agent is the variable α^i, inducing him a cost $c^i(\alpha^i) \geq 0$.

Direct contracting: the Principal offers a contract at time $t = 0$ for each Agent to incentivise them to act in her best interest at time $t = 1$, i.e. to improve the benefit of the firm.

Hierarchical contracting:

(i) the Principal offers a contract at time $t = 0$ for a designated Manager (Agent $i = 0$) to incentivise him to improve the benefit of the firm;

(ii) the Manager in turn offers contracts for the remaining Agents at time $t = 0$ and increases his own outcome by making an effort α^0 at time $t = 1$;

(iii) each Agent $i \in \{1, \dots, n\}$ makes an effort α^i at time $t = 1$ to increase his own outcome in exchange of the compensation.

- Interlinked Principal–Agent problems – Sequence of Stackelberg equilibria.

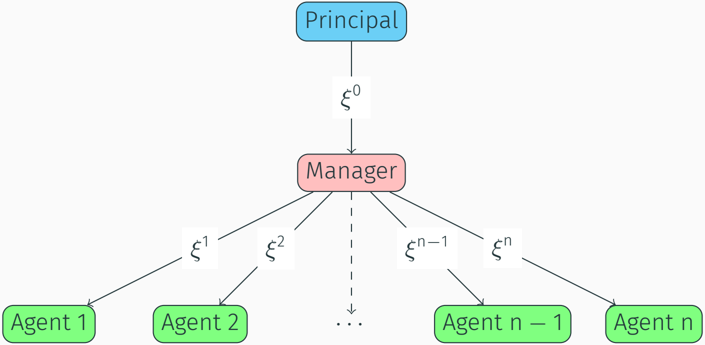

Figure: Sung's Model

Moral hazard in direct contracting: the Principal does not observe the effort α^i of the i−th Agent, she only observes his output X^i, for $i \in \{0, \dots, n\}$.

Moral hazard in this hierarchical contracting problem:

(i) the Manager does not observe the effort α^i of the i−th Agent, he only observes the output X^i, for $i \in \{1, \dots, n\}$.

- The contract ξ^i for the i−th Agent is indexed on X^i.

(ii) the Principal only observes the net benefit of the total hierarchy,

$$\zeta := \sum_{i=0}^{n} X^i - \sum_{i=1}^{n} \xi^i.$$

- The contract ξ^0 for the Manager is indexed on ζ.

In Sung (2015), the output processes are Gaussian, thus the monotone likelihood ratio is not bounded from below.

▶ No optimal contract in this case (see the existence of forcing contracts in Mirrlees (1999)).

But, in continuous–time, linear contracts are optimal in the case of drift control only (see Sannikov (2008)).

▶ It is common in one–period models to restrict the study to linear contracts:

$$\xi^i = \xi_0^i - \sup_{a \in \mathbb{R}} \left\{ aZ^i - c^i(a) \right\} + Z^i X^i + \frac{1}{2} R^i (Z^i)^2 \mathbb{V}\mathrm{ar}(X^i),$$

where Z^i is a parameter chosen by the Manager.

▶ Optimal effort: $\widehat{\alpha}^i(Z^i)$.

- The Manager controls the mean and the variance of his state variable ζ.

$$\zeta = \alpha^0 + \sigma^0 W^0 - \sum_{i=1}^{n} \left(\xi_0^i - \widehat{\alpha}^i(Z^i) + c^i(\widehat{\alpha}^i(Z^i)) + \frac{1}{2} R^i (Z^i \sigma^i)^2 \right) + \sum_{i=1}^{n} (1 - Z^i) \sigma^i W^i.$$

- The variance of ζ is not observed by the Principal, and thus the contract cannot be indexed on it.
- Sung (2015) restrict again the study to linear contracts, without the variance term.
- But, in continuous–time with volatility control, linear contracts are not optimal, see Cvitanić, Possamaï, and Touzi (2018)...

THE CONTINUOUS–TIME MODEL

Main contributions: Holmström and Milgrom (1987); Sannikov (2008).

Cvitanić, Possamaï, and Touzi (2018). General theory that allows to address a wide spectrum of Principal–Agent problems:

(i) identify a sub–class of contracts offered by the principal, which are revealing in the sense that the best–reaction function of the agent and his optimal control can be computed straightforwardly;

(ii) proving that the restriction is without loss of generality.

▶ Standard optimal control problem for the Principal.

▶ Extension to several Agents by Élie and Possamaï (2019); Baldacci, Possamaï, and Rosenbaum (2019), and even to a mean–field of Agents by Élie, Mastrolia, and Possamaï (2018); Élie, Hubert, Mastrolia, and Possamaï (2019).

▶ Only a few continuous–time models for hierarchical problems: Miller and Yang (2015), Li and Yu (2018).

The $i-$**th Agent**

- controls the drift of a process X^i with dynamic $dX^i_t = \alpha^i_t dt + \sigma^i dW^i_t$;
- receives a terminal payment ξ^i which is a function of $(X^i)_{t \in [0,1]}$.

The Manager

- controls the drift of a process X^0 with dynamic $dX^0_t = \alpha^0_t dt + \sigma^0 dW^0_t$;
- designs the contracts ξ^i for $i \in \{1, \ldots, n\}$;
- receives a terminal payment ξ^0.

The Principal only observes in continuous–time the process ζ

$$\zeta_t = \sum_{i=0}^{n} X^i_t - \sum_{i=1}^{n} \xi^i_t,$$

for $t \in [0, 1]$, and indexes the contract ξ^0 for the Manager on it.

The $i-$th Agent:

$$V_0^i(\xi^i) := \sup_{\alpha^i} \mathbb{E}^{\mathbb{P}^i}\left[-\exp\left(-R^i\left(\xi^i - \int_0^1 c^i(\alpha_t^i)\mathrm{d}t\right)\right)\right].$$

We will assume for simplicity that $c^i(a) = a^2/2k^i$ (quadratic costs).

The Manager:

$$V_0^0(\xi^0) := \sup_{\alpha^0,(\xi^i)_{i=1,\dots,n}} \mathbb{E}^{\mathbb{P}^0}\left[-\exp\left(-R^0\left(\xi^0 - \int_0^1 c^0(\alpha_t^0)\mathrm{d}t\right)\right)\right]$$

The Principal:

$$V_0 = \sup_{\xi^0} \mathbb{E}^{\mathbb{P}^\star}\left[\zeta_1 - \xi_1^0\right].$$

Assumption: the compensation for the i–th Agent can only be indexed on his own outcome process X^i.

▶ The **optimal** form of contracts for the i–th Agent is (see Sannikov (2008)):

$$\xi^i = \xi_0^i - \int_0^1 \mathcal{H}^i(Z_s^i)\mathrm{d}s + \int_0^1 Z_s^i \mathrm{d}X_s^i + \frac{1}{2}R^i \int_0^1 (Z_s^i)^2 \mathrm{d}\langle X^i \rangle_s, \tag{1}$$

where

(i) Z^i is a payment rate chosen by the Manager;

(ii) $\mathcal{H}^i(z) = \sup_{a\in\mathbb{R}}\{az - c^i(a)\}$ is the i–th Agent's Hamiltonian.

▶ The optimal effort of the i–th Agent is $\widehat{\alpha}_t^i = k^i Z_t^i$, and we can compute the dynamics of X^i and ξ^i with this optimal effort.

The Manager controls α^0 and Z^i, for $i \in \{1, \dots, n\}$.

Assumption: the Principal only observes ζ in continuous–time, where:

$$\mathrm{d}\zeta_t = \alpha_t^0 \mathrm{d}t + \sigma^0 \mathrm{d}W_t^0 + \sum_{i=1}^n \left(k^i Z_t^i - \frac{1}{2}(Z_t^i)^2 \left(k^i + R^i (\sigma^i)^2\right)\right)\mathrm{d}t$$
$$+ \sigma^i \sum_{i=1}^n (1 - Z_t^i)\mathrm{d}W_t^i,$$

and thus its quadratic variation (see Bichteler (1981)).

- The Manager controls the volatility of his state variable ζ.
- By Cvitanić, Possamaï, and Touzi (2018), the **optimal** form of contracts is:

$$\xi^0 = \xi_0^0 - \int_0^1 \mathcal{H}^0(Z_s, \Gamma_s)\mathrm{d}s + \int_0^1 Z_s \mathrm{d}\zeta_s + \frac{1}{2}\int_0^1 (\Gamma_s + R^0 Z_s^2)\mathrm{d}\langle\zeta\rangle_s. \qquad (2)$$

- Considering contract of the form (2), we can easily solve the Manager's problem by maximising his Hamiltonian:
  (i) the optimal effort on the drift is $\alpha^0_t := k^0 Z_t$;
  (ii) the optimal control on the i–th Agent's compensation is

$$Z^i_t := \frac{k^i Z_t - \left(\sigma^i\right)^2 \Gamma_t}{\left(k^i + R^i\left(\sigma^i\right)^2\right) Z_t - \left(\sigma^i\right)^2 \Gamma_t}.$$

- We can the compute the dynamics of ζ and ξ^0 under optimal efforts.

The Principal's problem is reduced to

$$V_0 = \sup_{(Z,\Gamma)\in\mathbb{R}^2} \mathbb{E}^{\mathbb{P}^0}\left[\zeta_T - \xi_T^0\right].$$

- The optimal payment rates for the Manager are given by the constant processes Z and $\Gamma := -R^0 Z^3$, where Z is solution of a well–posed maximisation problem.
- The optimal Γ is different from Sung (2015) where he forced $\Gamma = -R^0 Z^2$.
- We can write the optimal contracts designed by the Principal to the Manager, and by the Manager to each Agent.

NUMERICAL RESULTS

INCREASE THE MANAGER'S EFFORT...

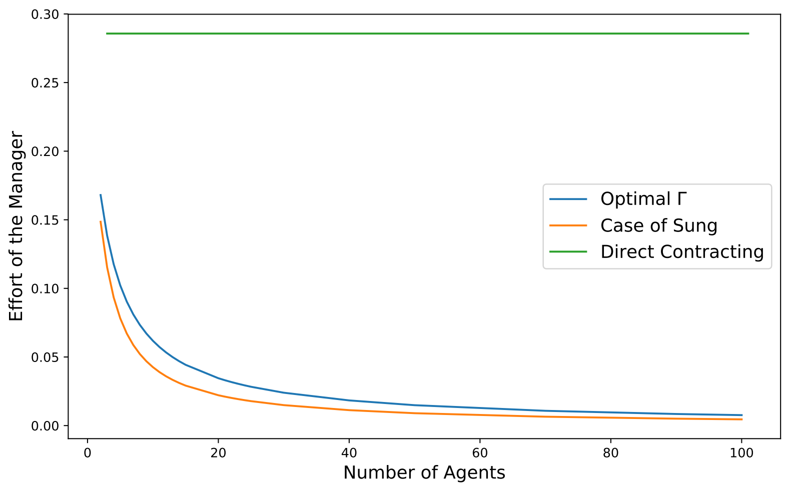

Figure: Effort of the Manager depending on the number of Agents.

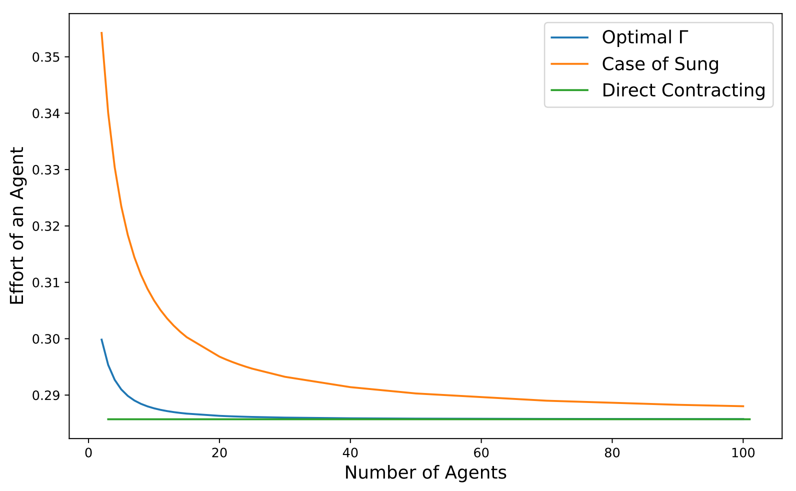

Figure: Effort of an Agent depending on the number of Agents.

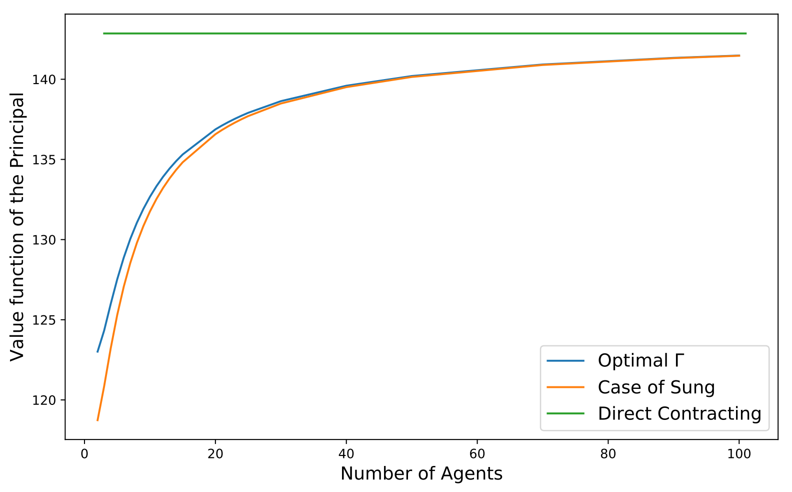

Figure: Value function of the Principal depending on the number of Agents.

CONCLUSION AND EXTENSIONS

- We improve the results of Sung (2015) by moving to continuous–time, since it allows to add a quadratic variation term in the contract for the Manager.

- This model can be extended to

(i) a more general hierarchy;

(ii) other forms of reporting ζ;

(iii) adding an "ability" parameter of the Manager.

- Extend to a more general model (work in progress) with:

(i) general output dynamics;

(ii) general utility functions;

(iii) general cost functions;

(iv) general form of reporting ζ.

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